The algorithm for the analysis of combined chaotic-stochastic processes

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Abstract

Many physical, chemical or sometimes financial phenomena are considered as being only chaotic, or purely stochastic. However a deeper understanding of the inherent nature of these processes sometimes exhibits both deterministic and stochastic features. The original idea of the paper is to find new models taking into account both behaviors, stochastic and chaotic, in order to understand and predict better the real physical phenomena, but also to model data for different applications such as biomedical or financial processes. The hypothesis about the approximation of the real data by fractional Brownian motion has been validated based on statistics, and the estimation of the Hurst exponent successfully characterized the agressiveness of the chaotic component.

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1 Introduction

The understanding of the dynamic behavior in real physical or industrial system is of almost importance, for analysis, synthesis, prediction, etc. It is sensible to consider that the behavior of many physical systems like phytoplankton, solar activity, oscillation of waves is a combination between chaotic or stochastic processes, which can be successfully used for prediction of health applications, meteorological phenomena etc [1, 2]. Many physical/ chemical or sometimes financial phenomena are considered as being only chaotic ((ex. Belousov - Zhabotinsky reaction) or purely stochastic (stock model price, integral Ito, Black-Scholes model), but in fact they could be both deterministic and stochastic [3, 4, 7]. So it is of utmost interest to find new models taking into account both behaviors, stochastic and chaotic, to understand and predict better the real physical phenomena, but also to model data for biomedical applications like (ECG, IRM).

The original idea in this paper is to juxtapose methods from stochastic signal analysis (nonstationary Gaussian processes, statistics from limit theorems by Nordin, Hurst exponent), and nonlinear (chaotic) dynamical system analysis (phase portrait, phase delayed plot, Lyapunov exponents), to develop a common methodology to analyze complex time series [5, 6, 8, 9]. Assuming that these two behaviors are inherently correlated, we are analyzing if there exists a correlation exists between the stochastic quantifiers (Hurst exponent, Garch method,ARMA) and chaotic quantifiers (Lyapunov exponents) [10, 11, 12]. To do that, different kind of stochastic-chaotic mixed processes shall be modeled and analyzed from different points of view to be developed. The idea is to find new models taking into account both behaviors, stochastic and chaotic, to understand and predict better the real physical phenomena, but also to model data for different applications such as biomedical or financial processes.

2 Proposed methodology

As classical approach, we assume a priory stochastic nature of time series model and construct a mathematical model as a random process [**?**]. Hurst exponent is defined like the estimate $\hat{H}$ of approximated fractional Brownian motion for these time series. On the other hand, for some deterministic systems, where the state is the solution of nonlinear differential or difference equation like $x_n = X(t_n)$, the behavior can be highly irregular and extremely complex. In some cases the behavior is estimated like chaotic. In the first approximation, we can determine the chaocity by the property of the system to construct it's trajectories in a bounded domain of the phase space. Properties of dynamical systems which generate chaotic solutions, have been widely discussed by the authors (results and references in the monographs [13, 14]). The simplest example is an one-dimensional dynamical system $x_{n+1} = f(x_n, \mu)$ which generates chaotic solution for some functions f and values of parameter μ . In particular, for logistic function f such as $x_{n+1} = \mu x_n(1 - x_n)$, the plot of solution looks like white noise with some values $\mu > 3.6$. So, the problem statement the nature of time series analysis nature is do the observed data have stochastic nature, or deterministic. Let $B\left(t\right)$, $0 \le t \le 1$ be a fractional Brownian motion with Hurst exponent H. Let's consider the normalized increments

$$\xi_k = n^H \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right) \sim \aleph\left(0;1\right) \ .$$

In the series of papers some limit theorems for the functions of these increments were proven. Let's denote

$$\alpha_k = n^H B\left(\frac{k}{n}\right) = \sum_{j=1}^{k-1} \xi_j \ .$$

There is a mean-square convergence:

$$\begin{array}{c} \dfrac{1}{n}\sum\limits_{k=1}^{n} \alpha_k \xi_k^3 \to -\dfrac{3}{2} \ , \quad H \in \left(0;\dfrac{1}{2}\right) \ , \\ \dfrac{1}{n^{1+H}} \sum \alpha_k^2 \xi_k^3 \to 3\eta \ , \quad H \in \left(0;\dfrac{1}{2}\right) \ , \end{array} \tag{1}$$

where

$$\eta \sim \aleph\left(0;\frac{1}{2H+2}\right) \ ;$$

$$\frac{1}{n^{2H}} \sum_{k=1}^{n} \alpha_k \xi_k^3 \to \frac{3}{2} B^2\left(1\right) \ , \quad H \in \left(\frac{1}{2};1\right) \ .$$

Let's normalize the increments and assume:

$$z_k = \left(\hat{\sigma}\right)^{-1} n^H y_k = \frac{0.8}{R_{1n}} y_k \ .$$

We assume that the hypothesis T holds:

$$z_k = \xi_k = n^H \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right) \ . \tag{2}$$

Assume $v_k = \sum_{j=1}^{k-1} z_j$ and calculate the statistical indexes:

$$\begin{aligned} A_n &= \frac{1}{n}\sum v_k z_k^3 \ , \qquad H \in \left(0; \tfrac{1}{2}\right) \ ; \\ B_n &= \frac{1}{n^{1+H}}\sum v_k^2 z_k^3 \ , \quad H \in \left(0; \tfrac{1}{2}\right) \ ; \\ D_n &= \frac{1}{n^{2H}}\sum v_k z_k^3 \ , \quad H \in \left(\tfrac{1}{2}; 1\right) \ . \end{aligned} \tag{3}$$

If hypothesis T is true, then there is convergence:

$$A_n \to -1.5 \ ; \quad B_n \to 3\eta \ ; \quad D_n \to \frac{3}{2}B^2\left(1\right) \ .$$

The decision about the hypothesis T is taken by comparing the actual values of statistics with their limiting theoretical values. Let's determine the deviation from the limit value $\delta = |A_n + 1.5|$ for statistic A_n; the limit distribution functions for statistics B_n, D_n:

$$F_1\left(x\right) = P\left\{3\eta < x\right\} = \Phi\left(\frac{x}{3d}\right) \ , \quad F_2\left(x\right) = 2\Phi\left(\sqrt{\frac{2}{3}x}\right) - 1 \ , x > 0 \ ,$$

where Φ is Laplace function, $d = \left(2H+2\right)^{-0.5}$.

Hypothesis T is accepted, if

$$\delta < \beta_0 \ , |B_n| < \beta_1 \ , H < 0.5 \ ; \quad 0 < D_n < \beta_2 \ , H > 0.5 \ , \tag{4}$$

where β_1, β_2 are quantiles of distributions of F_1, F_2, corresponding to the selected level of significance $\alpha = 0.1$. Then,

$$\beta_1 = \frac{4.95}{\sqrt{2H+2}} \ , \quad \beta_2 = 4.08 \ .$$

3 Results and discussion

The rate of convergence of statistics to the limit has been tested by numerical experiment for the first example ("ideal case"):

$$z_k = \left(\hat{\sigma}^{-1}\right) n^H \left(X\left(k\right) - X\left(k-1\right)\right) \ ; \quad X\left(t\right) = \sigma B_H\left(t\right) \ ,$$

where the values of fractional Brownian motion were obtained by simulation. The values of the statistical indexes A_n, B_n, D_n are shown in Table 1.

Table 1: Values of statistical indexes

H		A_n	B_n	D_n	β_1
0.1	$n = 200$	-1.30	0.84		3.34
	$n = 1000$	-1.32	2.63		3.34
0.2	$n = 200$	-1.21	0.81		3.20
	$n = 1000$	-1.35	1.74		3.20
0.3	$n = 200$	-2.00	0.37		3.07
	$n = 1000$	-1.10	0.50		3.07
0.4	$n = 200$	-0.55	1.26		2.96
	$n = 1000$	-2.51	0.83		2.96
0.6	$n = 200$			1.75	
	$n = 1000$			1.03	
0.7	$n = 200$			1.23	
	$n = 1000$			0.67	
0.8	$n = 200$			1.05	
	$n = 1000$			0.52	
0.9	$n = 200$			0.48	
	$n = 1000$			0.04	

From Table 1, it follows that

$$|B_n| < \frac{4.95}{\sqrt{2H+2}} = \beta_B \ , \quad H < 0.5 \ ; \quad 0 < D_n < 4.08 = \beta_D \ , \quad H > 0.5 \ ,$$

and deviation δ, $H < 0.5$, is an increasing function of H (for $H = 0.4$, $\delta \approx 1$).

The second example is a deterministic logistic chaotic sequence $x_{k+1} = 4x_k\left(1 - x_k\right)$, $k = 1, \ldots, 1049$. By procedure of estimation, we obtained $\hat{H} = 0.15$. The statistical indexes are as follows: $A_n = 0.6 > 0$, $|B_n| = 1.9 > \beta_B = 0.08$. Hypothesis T is rejected.

The third example. Assume that the observed values are an additive mixture of the deterministic chaotic and random sequences:

$$x_k = u_k + av_k \ ,$$

where u_k are the values of a dynamical system, v_k are the values of a random process. Sequences $\{u_k\}$, $\{v_k\}$ are normalized by energy, therefore $\frac{1}{n}\sum u_k^2 = \frac{1}{n}\sum v_k^2 = 1$. Then, the value a determines stochastic share in the observed data. In the example, $u_k = 4u_{k-1}\,(1 - u_{k-1})$, $v_k = \sigma B_H\left(\frac{k}{n}\right)$.

The stochastic sequence v_k is generated with $H_{fBm} = 0.1$–0.9. Table 2 shows estimate $\hat{H}$ of mixture and values of statistical indexes.

Table 2: Statistical indexes of mixture ($a = 1$, $a = 2$, $n = 2000$)

H		$\hat{H}$	A_n	B_n	D_n	β_1
0.1	$a=1$	0.6	-1.94	-0.07	-0.43	2.77
	$a=2$	0.1	-1.60	-0.40	-697	3.34
0.2	$a=1$	0.15	-5.35	-15.3	-1095	3.26
	$a=2$	0.15	-3.19	-6.17	-652	3.26
0.3	$a=1$	0.6	-2.54	-0.12	-0.56	2.77
	$a=2$	0.2	-5.0	-9.20	-477	3.19
0.4	$a=1$	0.15	-4.50	-7.38	-920	3.26
	$a=2$	0.15	-2.33	-1.03	-475	3.26
0.6	$a=1$	0.6	-0.91	-0.01	-0.20	2.77
	$a=2$	0.15	-4.0	15.8	-813	3.26
0.7	$a=1$	0.6	-1.35	-0.03	-0.30	2.77
	$a=2$	0.1	-1.08	0.47	-470	3.34
0.8	$a=1$	0.6	-1.37	-0.03	-0.30	2.77
	$a=2$	0.6	-0.68	-0.01	-0.15	2.77
0.9	$a=1$	0.6	-1.45	-0.03	-0.32	2.77
	$a=2$	0.6	-1.91	-0.07	-0.42	2.77

The table data show the "aggressiveness" of the chaotic component in relation to stochastic for $H_{fBm} \geq 0.2$. Inequalities (4) are not satisfied for these values of fBm and character of the mixture determines the logistic sequence. The deviation of statistics from the limit values is the same as for the "pure" fractional Brownian motion (Table 1) (for $H_{fBm} = 0.1$). Persistence ($\hat{H} > 0.5$) of investigated time series ($D_n < \beta_2$) means it has stochastic nature; antipersistent ($\hat{H} = 0.1$–0.2, $A_n \approx A$, $|B_n| < \beta_1$) admits the existence of the chaotic component (values tend to revert to a mean: an increase is likely to be followed by a decrease and vice-versa).

4 Conclusion

The mixed chaotic-stochastic sequence has been successfully analyzed with statistical tools. The increments of the sequence have been approximated as a random variable, after pre-processing. The agressiveness of the chaotic component has been estimated for the case of anti-persistant fBm approximation, and the statistical indexes have been identified.

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